Linear Algebra I

09/10/2023

You are **NOT** allowed to use any type of calculators.

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1 Systems of linear equations $(1+6+3+(1+6+3)=20 \text{ pts})$

Let a be a scalar. Consider the following system of linear equations in the unknowns x, y, and z:

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= -2 \\ 4x + y + (a^2 - 14)z &= a + 6. \end{aligned}$$

(a) Write down the corresponding augmented matrix .

(b) By performing elementary row operations, put the augmented matrix into row echelon form.

(c) Determine all values of a so that the system is consistent.

(d) For $a = 5$,

   (i) determine the *lead* and *free* variables.

   (ii) put the augmented matrix into *reduced* row echelon form by performing elementary row operations.

   (iii) find the solution set.

2 Matrix multiplication $(5+10=15 \text{ pts})$

Let $A \in \mathbb{F}^{p\times q}$ and $B \in \mathbb{F}^{q\times p}$. Prove or disprove the statements:

(a) $\det(AB) = \det(BA)$

(b) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ (For $M \in \mathbb{F}^{m\times m}$, $\operatorname{tr}(M) := \sum_{k=1}^{m} [M]_{kk}$.)

3 Determinants (10 + 15 = 25 pts)

Let $M(n) \in \mathbb{R}^{n \times n}$ be given by

$$[M(n)]_{ij} = \begin{cases} 1 & \text{if } |i-j| \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

For instance,

$$M(5) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) Compute the determinant of $M(n)$ for $n \in \{1, 2, 3\}$.

(b) Find real numbers a, b such that $\det\big(M(n+2)\big) = a \det\big(M(n+1)\big) + b \det\big(M(n)\big)$ for all $n \geqslant 1$.

4 Nonsingular matrices (15 + 15 = 30 pts)

Let a, b, c, d be scalars and $M \in \mathbb{F}^{m \times m}$. Consider the matrix

$$N = \begin{bmatrix} aM & bM \\ cM & dM \end{bmatrix}.$$

(a) Show that N is nonsingular if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and M are nonsingular.

(b) Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and M are nonsingular. Find the inverse of N.

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10 pts free